Persistence and optimal stopping in stochastic logistic harvesting with multiplicative jumps

Marcen Laguna*

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Abstract

In this paper, I unify the treatment of two questions for the stochastic logistic model with proportional harvesting and multiplicative Poisson jumps: (i) when does the resource stock persist almost surely in $(0, \infty)$, and (ii) when does the optimal harvesting effort admit a threshold representation that survives the introduction of jumps? I show that the Feller persistence condition $r - qE > \sigma^2/2$ is robust to multiplicative jumps with arbitrary intensity and jump-size distribution: jumps preserve positivity algebraically and do not interact with the integrability condition of the scale function. For the optimal harvesting problem with a linear-in-effort payoff and a quadratic cost, we show that the boundary of the inactive region admits an explicit closed-form characterization from the first-order condition of the control, without requiring smooth pasting. The threshold is monotone in wealth. The results provide a unified mathematical foundation for stopping-time formulations of renewable-resource harvesting under environmental risk; they are applied in a companion paper on deforestation dynamics in Colombian municipalities (Laguna, 2026).

Keywords: stochastic logistic equation; jump-diffusion; Feller boundary classification; optimal stopping; Hamilton–Jacobi–Bellman variational inequality; renewable resource harvesting.

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1 Introduction

The stochastic logistic equation with proportional harvesting,

$$dX_t = X_t\big[(r - qE) - \tfrac{r}{K}X_t\big]\,dt + \sigma X_t\,\mathrm{d}B_t, \qquad X_0 = x_0 > 0, \tag{1}$$

has been the workhorse of stochastic harvesting theory since Beddington and May (1977) and Braumann (1985). Two of its core results — the persistence condition $r - qE > \sigma^2/2$ that separates extinction from indefinite survival, and the gamma-distributed stationary law of the surviving process — have been extended in many directions: time-varying parameters (Brites and Braumann, 2017), multidimensional analogues for multispecies fisheries, and optimal harvest policies under different forms of effort and quota controls.

A more recent line of work (Ryan and Hanson, 1986; Hanson, 2007; Wang and Wirjanto, 2016) replaces or augments the Brownian noise with a compound Poisson process, modeling discrete environmental disasters (droughts, fires, deforestation events, market shocks) that the diffusion alone cannot capture. The natural specification preserves positivity multiplicatively:

$$dX_t = X_t\big[(r - qE) - \tfrac{r}{K}X_t\big]\,dt + \sigma X_t\,\mathrm{d}B_t + X_{t^-}\big(e^{Z_{N_t}} - 1\big)\,\mathrm{d}N_t, \tag{2}$$

with N a Poisson process of intensity $\lambda \geq 0$ and $\{Z_n\}$ i.i.d. with density f on $\mathbb{R}$.

The question we address in this note is structural: how much of the analytical framework that works for (1) survives in (2)? Specifically, we are interested in three results that are central to applications:

*School of Economics, Universidad del Rosario, Bogotá, Colombia. Email: marcen.laguna@urosario.edu.co.

(i) the *persistence condition* that determines whether X_t remains a.s. in $(0, \infty)$;
(ii) the *characterization of the boundary* between active and inactive harvesting in the associated optimal control problem;
(iii) the *monotonicity* of the inactive region in the agent's wealth, when the control problem is embedded in a household consumption–savings framework.

For (i), Feller's classification via the scale function (Karlin and Taylor, 1981, Ch. 15) gives an exact characterization for the pure-diffusion case (1). The classification does not extend directly to jump-diffusions: jumps can "overshoot" boundaries that pure diffusions never cross (Hanson, 2007, §7.6). We show, however, that for the multiplicative jump specification in (2), the persistence condition is *unchanged*: $r - qE > \sigma^2/2$ remains necessary and sufficient regardless of the jump intensity λ or the jump-size distribution f (Theorem 3.1). The reason is structural: multiplicative jumps preserve positivity by algebra, and between jumps the process evolves as a pure diffusion subject to Feller's classical condition.

For (ii) and (iii), we consider an optimal control problem in which an agent with wealth a_t harvests the resource at effort rate $e_t \in [0, \bar{e}]$, receives a state-dependent net payoff $\pi(x, e) = (\theta_0 + \theta_1 x/K)\, e$, pays a quadratic effort cost $C(e) = c_1 e + c_2 e^2$, and consumes the proceeds. The standard tool for the resulting stopping problem is the Hamilton–Jacobi–Bellman variational inequality (HJBVI) (Bensoussan and Lions, 1982; Øksendal and Sulem, 2019b). We show that for this class of payoffs the boundary of the inactive region $\{e^* = 0\}$ admits a *closed-form characterization from the first-order condition alone*: it does not require smooth pasting, and the same formula applies for diffusion and jump-diffusion dynamics (Theorem 4.2). Under standard regularity assumptions on the value function, this boundary is monotone increasing in wealth (Theorem 4.4). The economic interpretation, developed in detail in Laguna (2026), is that wealthier agents stop harvesting at higher remaining resource levels.

The three results are not deep individually; together they constitute a self-contained mathematical foundation for a class of harvesting problems with both Brownian and jump risk. The principal pedagogical contribution is to make explicit that the appearance of jumps does not require redeveloping the boundary classification from scratch, and that the FOC-based characterization of the threshold sidesteps the loss of smooth pasting that the literature on jump-diffusions has identified as a technical obstacle (Øksendal and Sulem, 2019b, §3.4).

The rest of this paper is organized as follows. Section 2 fixes notation and recalls the generator and scale function for one-dimensional jump-diffusions. Section 3 states and proves Theorem 3.1. Section 4 introduces the optimal harvesting problem, states Theorems 4.2 and 4.4, and proves them. Section 5 sketches the application to the deforestation problem of Laguna (2026).

2 Preliminaries

2.1 The jump-diffusion process

Throughout, $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is a filtered probability space supporting a standard Brownian motion (B_t), a Poisson process (N_t) of constant intensity $\lambda \geq 0$ independent of B, and an i.i.d. sequence $(Z_n)_{n \geq 1}$ of $\mathbb{R}$-valued random variables with common density f, independent of B and N. We assume $\mathbb{E}[|Z_1|] < \infty$.

The resource process $X = (X_t)_{t \geq 0}$ solves

$$dX_t = X_t\big[\alpha - \beta X_t\big]dt + \sigma X_t\, dB_t + X_{t^-}\big(e^{Z_{N_t}} - 1\big)dN_t, \qquad X_0 = x_0 > 0, \tag{3}$$

where $\alpha = r - qE \in \mathbb{R}$, $\beta = r/K > 0$, and $\sigma > 0$.

Remark 2.1. For the deforestation interpretation, α, β, σ are fixed structural parameters with $\alpha > 0$ (gross growth exceeds harvest pressure on average), and $Z_n \leq 0$ a.s. (jumps model environmental disasters). The mathematics below does not require $\alpha > 0$ or $Z_n \leq 0$; we keep the general formulation.

The diffusion and jump coefficients in (3) are locally Lipschitz on $(0,\infty)$ and of linear growth, so (3) admits a unique strong solution up to the exit time $\tau_0 \wedge \tau_\infty$ of $(0,\infty)$ (Øksendal and Sulem, 2019b, Theorem 1.19). The persistence question of Section 3 is precisely whether $\tau_0 = \infty$ a.s.

The *infinitesimal generator* of X is

$$\mathcal{H}\varphi(x) = \underbrace{\mu(x)\,\varphi'(x) + \tfrac{1}{2}\sigma^2 x^2\,\varphi''(x)}_{\mathcal{L}\varphi(x)} + \underbrace{\lambda \int_{\mathbb{R}} \big[\varphi(xe^z) - \varphi(x)\big] f(z)\,dz}_{\mathcal{J}\varphi(x)}, \tag{4}$$

acting on $\varphi \in C^2((0,\infty))$ with bounded derivatives, where $\mu(x) = x(\alpha - \beta x)$ is the diffusion drift.

2.2 Scale function for the diffusion part

The Feller classification of the boundaries of the pure-diffusion process is the basis of our Theorem 3.1. Let

$$s(x) = \exp\!\left(-\int^x \frac{2\mu(y)}{\sigma^2 y^2}\,\mathrm{d}y\right) = x^{-\gamma}\,e^{2\beta x/\sigma^2}, \qquad \gamma \equiv \frac{2\alpha}{\sigma^2}, \tag{5}$$

be the scale density and $S(x) = \int^x s(y)\,\mathrm{d}y$ the scale function. The classical boundary criterion (Karlin and Taylor, 1981, Ch. 15) states that, for the pure-diffusion process ($\lambda = 0$ in (3)), the boundary at 0 is non-attainable in finite time if and only if $\int_0^c s(y)\,\mathrm{d}y = +\infty$ for some (equivalently, every) $c \in (0,\infty)$. Direct computation gives

$$\int_0^c s(y)\,\mathrm{d}y < \infty \iff \gamma < 1 \iff \alpha < \tfrac{\sigma^2}{2}. \tag{6}$$

We will not need this classification at $+\infty$, where $s(x) \to \infty$ exponentially and the boundary is always non-attainable.

3 Persistence under multiplicative jumps

3.1 Statement

Theorem 3.1. *Consider the jump-diffusion (3) with $\alpha, \beta > 0$, $\sigma > 0$, $\lambda \geq 0$, and an i.i.d. jump-size sequence (Z_n) with common density f on $\mathbb{R}$ satisfying $\mathbb{E}[|Z_1|] < \infty$. For every $x_0 > 0$:*

(i) If $\alpha > \sigma^2/2$, then $\mathbb{P}_{x_0}(X_t > 0 \text{ for all } t \geq 0) = 1$.

(ii) If $\alpha \leq \sigma^2/2$, then $\mathbb{P}_{x_0}(\inf_{t\geq 0} X_t = 0) > 0$.

In particular, the boundary 0 is non-attainable in finite time if and only if $\alpha > \sigma^2/2$, independently of λ and the law of Z_1.

Proof. The proof of (i) decomposes the trajectory of X into a deterministic alternation of pure-diffusion segments and jump epochs, and applies Feller's classical criterion on each segment. The proof of (ii) uses a comparison argument with the pure-diffusion case.

Proof of (i): sufficiency. Assume $\alpha > \sigma^2/2$. Let $0 = T_0 < T_1 < T_2 < \cdots$ denote the jump times of N, with the convention $T_0 = 0$, and set $T_\infty = \lim_n T_n$. By standard properties of the Poisson process, $T_n \to \infty$ a.s., so the number of jumps in any compact interval $[0,t]$ is a.s. finite. Define

$$\tau_0 \equiv \inf\{t \geq 0 : X_t = 0\}, \tag{7}$$

with $\inf \emptyset = +\infty$. We will show $\tau_0 = +\infty$ a.s.

We proceed by induction on the jump epochs. The induction hypothesis is:

$$\mathbb{P}_{x_0}\big(X_t > 0 \text{ for all } t \in [0, T_n]\big) = 1 \qquad \text{for all } n \geq 0. \tag{8}$$

Base case ($n = 0$): trivially, $X_0 = x_0 > 0$.

Inductive step. Assume (8) holds for some $n \geq 0$. Conditionally on $\mathcal{F}_{T_n}$ and on $\{X_{T_n} = y\}$ with $y > 0$, on the interval $[T_n, T_{n+1})$ the process X satisfies the pure-diffusion SDE

$$dX_t = X_t(\alpha - \beta X_t)\, dt + \sigma X_t\, dB_t, \qquad X_{T_n} = y, \tag{9}$$

because no Poisson jump occurs in the open interval (T_n, T_{n+1}). By the Feller classification (6), $\alpha > \sigma^2/2$ implies that the process (9) does not reach 0 in finite time a.s., starting from any $y > 0$. Hence

$$\mathbb{P}\big(X_t > 0 \text{ for all } t \in [T_n, T_{n+1}^-) \,\big|\, \mathcal{F}_{T_n}\big) = 1. \tag{10}$$

At the jump time T_{n+1}, the process makes a multiplicative jump:

$$X_{T_{n+1}} = X_{T_{n+1}^-} \cdot e^{Z_{n+1}}. \tag{11}$$

Since $e^z > 0$ for every $z \in \mathbb{R}$ and $X_{T_{n+1}^-} > 0$ by (10), we have $X_{T_{n+1}} > 0$ a.s. Combined with the induction hypothesis, this gives $X_t > 0$ for all $t \in [0, T_{n+1}]$, completing the step.

By induction, $X_t > 0$ for all $t \in [0, T_n]$ a.s. for every n. Since $T_n \to \infty$ a.s., this yields $X_t > 0$ for all $t \geq 0$ a.s., i.e., $\tau_0 = +\infty$ a.s.

Proof of (ii): necessity. Assume $\alpha \leq \sigma^2/2$. We construct an event of positive probability on which X hits 0.

Consider first the case $\lambda = 0$: X is a pure diffusion, and (6) gives $\int_0^c s(y)\, \mathrm{d}y < \infty$, so 0 is attainable in finite time with positive probability.

For $\lambda > 0$, consider the event

$$A_T \equiv \{N_T = 0\} = \{\text{no Poisson jumps occur in } [0, T]\}, \tag{12}$$

which has probability $e^{-\lambda T} > 0$. Conditionally on A_T, the process X evolves on $[0, T]$ as the pure diffusion (9) with $y = x_0$. By the previous paragraph, there exists $T_0 < \infty$ and an event $B \in \mathcal{F}_{T_0}$ with $\mathbb{P}(B) > 0$ on which the pure diffusion hits 0 before T_0. Therefore

$$\mathbb{P}\big(\inf_{t \geq 0} X_t = 0\big) \geq \mathbb{P}(A_{T_0} \cap B) > 0, \tag{13}$$

since A_{T_0} and B are independent (one depends on N and the other on B). □

Remark 3.2. Three comments on Theorem 3.1 are in order.

(a) The result identifies the persistence condition as a property of the diffusion component alone. The intuition is that the multiplicative structure of the jumps preserves positivity by algebra ($X_{T^+} = X_{T^-} \cdot e^Z > 0$), while between jumps the process is governed by (9), on which Feller's classification applies verbatim. The integrability of s near 0 is a property of the diffusion drift and volatility, not of the jump kernel.

(b) The contrast with additive jumps is worth stressing. For SDEs of the form $dX = \mu\, dt + \sigma\, dB + \int_{\mathbb{R}} z\, N(dt, dz)$ with Z unconstrained, jumps can drive X across the boundary 0 in a single event, and Theorem 3.1 fails. The multiplicative specification (3) sidesteps this issue, which is also the reason it is the canonical model for population biology and resource economics: it respects the positivity constraint by construction.

(c) The result does not require any moment condition on Z_1 beyond $\mathbb{E}[|Z_1|] < \infty$, and in fact Z_1 may have arbitrary support in $\mathbb{R}$. In particular, the deforestation paper's restriction $Z_n \leq 0$ (jumps are always disasters) is not needed for persistence.

4 Optimal stopping with linear payoff

4.1 The harvesting problem

We embed (3) in an optimal control problem. An agent with wealth a_t and access to the resource X_t chooses consumption $c_t \geq 0$ and harvesting effort $e_t \in [0, \bar{e}]$ to maximize

$$\mathbb{E}\int_0^\infty e^{-\rho t}\big[U(c_t) - C(e_t)\big]\,dt, \tag{14}$$

subject to

$$da_t = \big[ra_t + \pi(X_t, e_t) - c_t\big]\,dt, \qquad a_t \geq \underline{a}, \tag{15}$$

$$dX_t = \mu_e(X_t, e_t)\,dt + \sigma X_t\,dB_t + X_{t^-}(e^{Z_{N_t}} - 1)\,dN_t, \tag{16}$$

with controlled drift $\mu_e(x, e) = x[(r - qe) - (r/K)x]$, state-dependent linear payoff

$$\pi(x, e) = \big(\theta_0 + \theta_1\, x/K\big)\, e, \qquad \theta_1 > 0,\ \theta_0 \in \mathbb{R}, \tag{17}$$

and quadratic effort cost $C(e) = c_1 e + c_2 e^2$ with $c_1, c_2 > 0$. The utility U is CRRA: $U(c) = c^{1-\gamma}/(1-\gamma)$ with $\gamma > 0$, $\gamma \neq 1$ (with the logarithmic case as the limit).

Assumption 4.1. $\rho > r > 0$, $\gamma > 0$, and $\alpha = r - q\bar{e} > \sigma^2/2$ (so that persistence holds for any admissible policy; cf. Theorem 3.1).

Let $V \in C^2((\underline{a}, \infty)\times(0, K))$ denote the value function. The associated Hamilton–Jacobi–Bellman equation is

$$\rho V = \sup_{c\geq 0,\, e\in[0,\bar{e}]} \big\{U(c) - C(e) + V_a\big[ra + \pi(x, e) - c\big] + V_x \mu_e(x, e) + \tfrac{1}{2}\sigma^2 x^2 V_{xx} + \mathcal{J}V\big\}, \tag{18}$$

with $\mathcal{J}V$ the jump operator from (4). The wealth dynamics are deterministic in this formulation; the only stochastic state is X, which is why the jump and diffusion operators act on V only through its x-derivatives.

4.2 Threshold characterization

The first-order condition (FOC) for e in (18), ignoring the constraint $e \in [0, \bar{e}]$, is

$$V_a\,(\theta_0 + \theta_1\, x/K) - c_1 - 2c_2 e - V_x\, qx = 0, \tag{19}$$

which gives the unconstrained maximizer

$$e^\circ(a, x) \equiv \frac{V_a(a, x)(\theta_0 + \theta_1\, x/K) - c_1 - V_x(a, x)\, qx}{2c_2}. \tag{20}$$

The constrained optimum is $e^*(a, x) = \max\{0, \min\{\bar{e}, e^\circ(a, x)\}\}$.

Define the *inactive region*

$$\mathcal{I} \equiv \big\{(a, x) \in (\underline{a}, \infty) \times (0, K) : e^*(a, x) = 0\big\}. \tag{21}$$

Theorem 4.2. *Let $V \in C^2((\underline{a}, \infty) \times (0, K))$ solve (18) under Assumption 4.1, with $V_a > 0$ on the domain. Then:*

(i) $(a, x) \in \mathcal{I}$ if and only if

$$V_a(a, x)\big(\theta_0 + \theta_1\, x/K\big) - c_1 - V_x(a, x)\, qx \leq 0. \tag{22}$$

(ii) Equivalently, $(a,x) \in \mathcal{I}$ if and only if $x/K \le S^*(a,x)$, where

$$S^*(a,x) \equiv \frac{c_1 + V_x(a,x)\, qx - V_a(a,x)\, \theta_0}{V_a(a,x)\, \theta_1}. \tag{23}$$

(iii) The boundary $\partial\mathcal{I} \cap [(\underline{a},\infty) \times (0,K)]$ is the level set $\{(a,x) : S = S^*(a,x)\}$ where $S = x/K$, and the same formula (23) applies for every $\lambda \ge 0$.

Proof. The maximand of (18) as a function of e alone is a concave quadratic with leading coefficient $-2c_2 < 0$:

$$e \mapsto -c_2 e^2 + \big[V_a(\theta_0 + \theta_1 x/K) - c_1 - V_x qx\big]\, e + (\text{terms independent of } e).$$

The unconstrained maximum (20) is interior if and only if $e^\circ > 0$, which by inspection is equivalent to the strict version of (22). The corner solution $e^* = 0$ is optimal precisely when $e^\circ \le 0$, which is (22) with the inequality non-strict. This proves (i).

Item (ii) follows from (i) by algebra: rearranging (22) for x/K, using $\theta_1 > 0$, gives $x/K \le (c_1 + V_x qx - V_a\theta_0)/(V_a\theta_1) = S^*(a,x)$.

For (iii), the boundary $\partial\mathcal{I}$ is where the inequality in (22) is tight. The formula (23) depends on V and its partials, which in turn depend on λ through the equation (18); however, the *functional form* of (23) is the same regardless of λ. This is the substance of the claim: the threshold formula is structural, not specific to the diffusion case. □

Remark 4.3 (The role of smooth pasting). In the optimal stopping literature on jump-diffusions, smooth pasting is known to fail at boundaries that the process can cross discontinuously (Øksendal and Sulem, 2019b, §3.4), (Hanson, 2007, §7.6). The characterization in Theorem 4.2 bypasses this difficulty: the inactive region is defined directly by the FOC of the control, and (23) is obtained without any pasting condition. The HJBVI complementarity formulation (Bensoussan and Lions, 1982), in which the value function is recovered by solving

$$\min\big\{\rho V - \mathcal{H}V,\; V - V^*\big\} = 0,$$

with V^* the value of permanent inaction, is consistent with (23) on the part of the boundary where the FOC binds. The characterization in Theorem 4.2 identifies the boundary of the region in which the instantaneous optimal control satisfies $e^*(a,x) = 0$. This should be distinguished from the stopping region of the associated HJB variational inequality, namely the set where the value of continuation coincides with the value of permanent inaction $\mathcal{S} = \{(a,x) : V(a,x) = V^*(a,x)\}$. In general, these two objects need not coincide: even when the optimal current effort is zero, continuation may still retain positive option value because the state process can later return to regions where harvesting becomes profitable. This distinction is particularly relevant for renewable-resource dynamics such as (16), since under the policy $e \equiv 0$ the stock regenerates stochastically and trajectories starting in the inactive region need not remain there. Accordingly, formula (23) should be interpreted as the boundary of the zero-effort policy region determined by the first-order condition, rather than as a smooth-pasting or value-matching boundary. Under additional invariance conditions on the inactive region, the two notions coincide, but such conditions are not imposed here.

4.3 Monotonicity

Theorem 4.4. *Under the hypotheses of Theorem 4.2, assume in addition that on $\mathcal{I}^c$ the value function satisfies:*
(M1) $V_{aa}(a,x) < 0$ (strict concavity in wealth);
(M2) $V_a(a,x) > 0$ and $\theta_0 + \theta_1\, x/K > 0$ on the boundary $\partial\mathcal{I}$;
(M3) $V_{ax}(a,x) \ge 0$ (wealth and resource are complements in the value function).

Then the boundary $\partial\mathcal{I}$ can be parametrized as the graph of a function $a \mapsto S^*(a) \in [0,1]$ defined implicitly by (23), and this function is non-decreasing in a.

Proof. On $\partial\mathcal{I}$, equation (22) holds with equality:

$$V_a(a,x)\big(\theta_0 + \theta_1\, x/K\big) - c_1 - V_x(a,x)\, qx = 0. \tag{24}$$

Differentiating implicitly with respect to a, treating x as $x = KS^*(a)$:

$$V_{aa}(\theta_0 + \theta_1 S^*) + V_a\, \theta_1 \frac{dS^*}{da} - V_{xa}\, qx = 0. \tag{25}$$

Solving (25) for dS^*/da:

$$\frac{dS^*}{da} = \frac{V_{xa}\, qx - V_{aa}(\theta_0 + \theta_1 S^*)}{V_a\, \theta_1}. \tag{26}$$

The denominator $V_a\,\theta_1 > 0$ by (M2). For the numerator: $V_{xa}\, qx \geq 0$ by (M3) and $qx > 0$; and $-V_{aa}(\theta_0 + \theta_1 S^*) > 0$ by (M1) and (M2). Hence $dS^*/da \geq 0$. □

Remark 4.5. Condition (M3) is non-trivial. It states that the cross-derivative V_{ax} is non-negative, which is the statement that the marginal value of wealth is increasing (or at least non-decreasing) in the resource stock. Economically this is natural: a wealthier agent values the option to harvest more highly when more resource is available. For the deforestation problem, the companion paper verifies (M3) numerically but does not prove it analytically. A proof would require either (a) explicit construction of V in a tractable special case, or (b) propagation of monotonicity through the HJB iteration. This is a natural target for an extension or follow-up note.

5 Application: deforestation in Colombian municipalities

The three results above are applied in Laguna (2026) to a structural model of land-use change in Colombian municipalities over the period 2001–2022. In that model, the resource X_t represents the remaining forest stock of a municipality, the effort e_t is the rate of forest conversion to productive land, and the payoff $\pi(X, e)$ matches a reduced-form regression in which the marginal returns to deforestation depend linearly on remaining forest cover.

The empirical analysis estimates $\hat{\theta}_0 = -0.067$ and $\hat{\theta}_1 = 0.14$, giving an implied threshold $S^* = -\hat{\theta}_0/\hat{\theta}_1 \approx 0.48$ at which the marginal returns to conversion change sign. The structural model reproduces this threshold endogenously through formula (23), with the wealth dimension generating cross-sectional heterogeneity in stopping behavior (Theorem 4.4): wealthier municipalities, proxied by their fiscal capacity and asset endowment, cease deforestation at higher remaining forest shares.

The jump component (3) captures discrete environmental disasters — droughts, wildfires, and the rapid forest loss that followed the 2016 Colombian peace agreement (Prem et al., 2020). The persistence result (Theorem 3.1) ensures that the SDE has a well-defined solution in $(0, K)$ for all admissible harvest policies, and that the stationary distribution implied by the model is supported on the interior. Without the structural argument of Theorem 3.1, the introduction of jumps would have required a separate proof of well-posedness; with it, the diffusion-based persistence theory of Braumann (1985) extends to the empirically relevant jump-diffusion specification at no analytical cost.

The characterization in Theorem 4.2 plays a central computational role in the companion deforestation paper. Rather than solving a free-boundary problem through smooth-pasting conditions, the empirical implementation computes the zero-effort policy boundary directly from the first-order condition (23). This is particularly useful in the jump-diffusion setting, where smooth pasting may fail because trajectories can cross boundaries discontinuously. The FOC-based characterization therefore provides a numerically tractable way to recover threshold policies while remaining consistent with the HJB variational-inequality framework.

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Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Data availability

No data was used for the research described in the article.

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