From equilibria to stationary distributions:
a dictionary between deterministic dynamics and one-dimensional diffusions process, with application to logistic harvesting

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May 2026

Resumen

I establish an operational dictionary between the qualitative analysis of one-dimensional ordinary differential equations and that of one-dimensional diffusion processes. The notions of fixed point, linear stability, and basin of attraction in the deterministic framework admit precise counterparts in the stochastic framework: stationary distribution, spectrum of the Fokker–Planck operator, and Feller's boundary classification. I articulate this correspondence as a four-step procedure starting from the coefficients (μ, Σ) of the SDE. As an illustration, we apply the framework to the logistic model with proportional harvesting under multiplicative Brownian perturbation: I recover the persistence condition $r - qE > \sigma^2/2$, identify the stationary distribution as a Gamma distribution, and derive the stochastic correction to the maximum sustainable yield.

Keywords: diffusion processes, Fokker–Planck equation, Feller's boundary classification, stochastic logistic model, optimal harvesting.

1. Introduction

The transition from deterministic to stochastic analysis of dynamical systems is usually presented as a conceptual leap: instead of deterministic trajectories converging to a fixed point one obtains processes whose distributions converge to an invariant measure. The underlying formalism, however, is closely parallel. In one dimension, the central objects of the qualitative analysis of ODEs — fixed points, linear stability, basins of attraction — have explicit counterparts in diffusion theory, all computable from the coefficients of the SDE through a single procedure.

This note has two purposes. First, to articulate this parallelism as an operational dictionary (Section 2 and Figure 1), emphasizing that the technical tools — scale function, Feller's criterion, Fokker–Planck equation — are the *exact analogs* of the objects of phase analysis. Second, to illustrate its usefulness by applying it to the logistic model with proportional harvesting (Section 3), a classical biomathematical example where noise produces non-trivial quantitative effects: it biases the mean abundance, modifies the optimal effort, and introduces new extinction conditions. The results in Section 3 are scattered across Beddington & May [1], Braumann [2], and Brites–Braumann [3]; my contribution is to present them as a direct application of the general scheme rather than as *ad hoc* computations.

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2. General framework: one-dimensional diffusions

2.1. Coefficients, generator, and the key quantity

Let $X(t)$ be a diffusion process on an interval $I = (\ell, r_b) \subseteq \mathbb{R}$, defined by the Itô SDE

$$dX(t) \;=\; \mu(X(t))\,dt \;+\; \Sigma(X(t))\,\mathrm{d}W_t, \qquad X(0) = x_0 \in I, \tag{1}$$

with $\mu, \Sigma \in C^1(I)$ and $\Sigma(x) > 0$ on I. The *infinitesimal generator* is

$$\mathcal{L} \;=\; \mu(x)\,\partial_x + \tfrac{1}{2}\Sigma^2(x)\,\partial_{xx},$$

and its formal adjoint, the *Fokker–Planck operator*, acts on densities as $\mathcal{L}^* p = -\partial_x(\mu p) + \frac{1}{2}\partial_{xx}(\Sigma^2 p)$. The transition density $p(x, t \mid x_0)$ satisfies $\partial_t p = \mathcal{L}^* p$.

All the qualitative information we need is encoded in the ratio

$$\kappa(x) \;\equiv\; \frac{2\mu(x)}{\Sigma^2(x)}, \tag{2}$$

and in its indefinite integral $\int^x \kappa(y)\,\mathrm{d}y$. The next subsections show how, starting from κ, one obtains in this order (i) hitting probabilities, (ii) boundary classification, and (iii) the stationary density.

2.2. Scale function, hitting, and boundary classification

For $\ell < a < x < b < r_b$, let $u(x) = \Pr\{\tau_b < \tau_a \mid X_0 = x\}$ where τ_a, τ_b are the hitting times of a and b. Applying Dynkin's formula at the stopping time $\tau = \tau_a \wedge \tau_b$ yields that u solves the boundary value problem

$$\mathcal{L}u(x) = 0 \quad \text{on } (a, b), \qquad u(a) = 0,\; u(b) = 1. \tag{3}$$

The equation $\mathcal{L}u = 0$ is linear of second order and can be integrated explicitly: setting $v = u'$ and separating variables, $v(x) \propto \exp(-\int^x \kappa)$. This motivates the following definition.

Definición 2.1 (Scale function)**.** The *scale density* and the *scale function* of (1) are

$$s(x) \;=\; \exp\Big(-\int^x \kappa(y)\,\mathrm{d}y\Big), \qquad S(x) \;=\; \int^x s(y)\,\mathrm{d}y.$$

Imposing the boundary conditions in (3) yields the classical formula [5, Ch. 15]

$$u(x) \;=\; \frac{S(x) - S(a)}{S(b) - S(a)}. \tag{4}$$

Letting $a \to \ell^+$ with b fixed gives immediately:

Lema 2.2 (Feller's boundary criterion)**.** *Fix $c \in I$. The boundary ℓ is* non-attainable *in finite time (almost surely, for every $x_0 \in I$) if and only if*

$$\int_\ell^c s(x)\,\mathrm{d}x \;=\; +\infty. \tag{5}$$

An analogous statement holds for r_b with $\int_c^{r_b} s(x)\,\mathrm{d}x$.

When both integrals diverge, (1) admits a unique strong solution that remains in I for all $t \geq 0$. This is the stochastic replacement of the deterministic notion of domain invariance.

2.3. Stationary density

If both boundaries are non-attainable, $X(t)$ is confined to I and may admit a stationary density p^* characterized by $\mathcal{L}^* p^* = 0$. Integrating $\mathcal{L}^* p^* = 0$ once with vanishing probability current

$$J \;=\; \mu(x)\, p^*(x) - \tfrac{1}{2}\, \partial_x\big[\Sigma^2(x)\, p^*(x)\big] \;=\; 0$$

and separating variables leads to:

Lema 2.3 (Stationary density). *The stationary density of* (1)*, when it exists, has the form*

$$p^*(x) \;=\; \frac{C}{\Sigma^2(x)} \exp\Big(\int^x \kappa(y)\,\mathrm{d}y\Big), \qquad x \in I, \tag{6}$$

with $C > 0$ *the normalization constant. Equivalently,* $p^*(x) \propto m(x)$*, where* $m(x) = 1/\big(\Sigma^2(x)\, s(x)\big)$ *is the* speed measure *of the process.*

The existence of p^* requires the right-hand side of (6) to be integrable on I. This condition replaces the deterministic criterion of existence of an interior fixed point, and in fact coincides with it in the limit $\Sigma \to 0$ in a sense made precise below.

2.4. The dictionary

The deterministic equation $\dot{x} = f(x)$ corresponds to (1) with $\mu = f$ and $\Sigma \equiv 0$. The two evolution equations — $\dot{x} = f(x)$ and $\partial_t p = \mathcal{L}^* p$ — share structure: both are linear evolution equations in their respective spaces (points of the interval I vs. densities on I), and their stationary states are obtained by setting the right-hand side to zero. Table 1 and Figure 1 make this correspondence explicit.

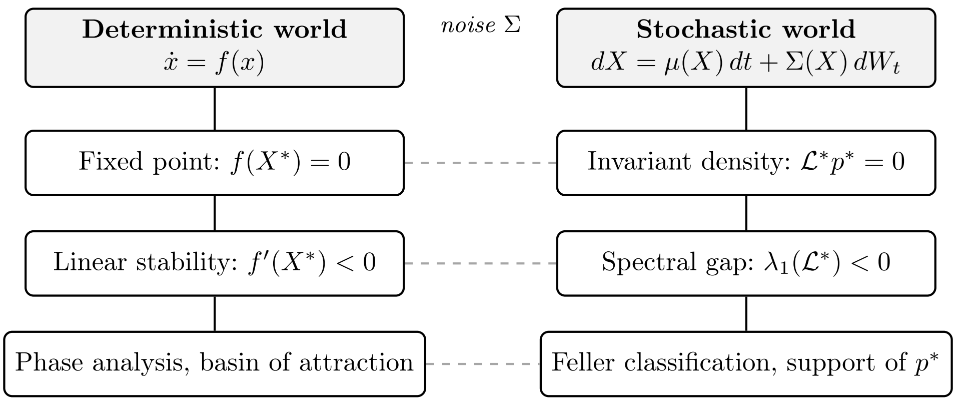

Figura 1: Dictionary between one-dimensional deterministic dynamics and one-dimensional diffusion processes. Vertical arrows show the flow of analysis within each world; dashed horizontal arrows mark the conceptual correspondence between analogous objects.

3. Application: stochastic logistic model with proportional harvesting

3.1. Deterministic baseline

Let $X(t)$ be the biomass of a population exploited at constant effort $E \geq 0$ with catchability coefficient $q > 0$, so that the removal rate is qEX. The classical logistic model reads

$$\dot{X} \;=\; X\big[(r - qE) - \tfrac{r}{K} X\big]\,, \tag{7}$$

Cuadro 1: Correspondence of objects. The stochastic column reduces to the deterministic one in the limit $\Sigma \to 0$.

	ODE: $\dot{x} = f(x)$	**SDE:** $dX = \mu\, dt + \Sigma\, dW$
State	point $x \in I$	measure $p(\cdot, t)$ on I
Evolution	$\dot{x} = f(x)$	$\partial_t p = \mathcal{L}^* p$
Stationary state	$f(X^*) = 0$	$\mathcal{L}^* p^* = 0$, eq. (6)
Local stability	sign of $f'(X^*)$	spectral gap of $\mathcal{L}^*$
Convergence rate	$e^{f'(X^*)t}$	$e^{\lambda_1 t}$, $\lambda_1 < 0$
Domain invariance	f points inward	$\int_\ell s = \int^{r_b} s = +\infty$ (Lem. 2.2)
Basin of attraction	phase analysis	support of p^*

with $r > 0$ the intrinsic growth rate and $K > 0$ the carrying capacity. The equilibria are $X_1^* = 0$ and $X_2^* = K(1 - qE/r)$. The deterministic persistence condition is $qE < r$; in that case X_2^* is a global attractor on $(0, \infty)$, and maximizing the stationary yield $Y(E) = qE\, X_2^*$ produces the classical maximum sustainable yield (MSY)

$$E_{\text{MSY}} = \tfrac{r}{2q}, \qquad Y_{\text{MSY}} = \tfrac{rK}{4}, \qquad X^*_{\text{MSY}} = \tfrac{K}{2}. \tag{8}$$

3.2. Stochastic model and application of the framework

Following [1, 2], we perturb the per-capita growth rate by Gaussian white noise of intensity $\sigma > 0$, leading to the Itô SDE

$$dX \;=\; X\big[(r - qE) - \tfrac{r}{K}X\big]\, dt \;+\; \sigma X\, dW_t, \qquad X(0) = x_0 > 0. \tag{9}$$

This fits framework (1) on $I = (0, \infty)$ with coefficients

$$\mu(x) = x\big[(r - qE) - \tfrac{r}{K}x\big]\,, \qquad \Sigma^2(x) = \sigma^2 x^2. \tag{10}$$

We apply the procedure of Section 2.

Step 1 (κ and its integral). From (10),

$$\kappa(x) \;=\; \frac{2\mu(x)}{\Sigma^2(x)} \;=\; \frac{2(r - qE)}{\sigma^2} \cdot \frac{1}{x} \;-\; \frac{2r}{\sigma^2 K}, \tag{11}$$

so that $\int^x \kappa(y)\, \mathrm{d}y = \gamma \ln x - \lambda x + \text{const}$, where

$$\gamma \;\equiv\; \frac{2(r - qE)}{\sigma^2}, \qquad \lambda \;\equiv\; \frac{2r}{\sigma^2 K}. \tag{12}$$

Step 2 (scale function and boundaries). The scale density is $s(x) = x^{-\gamma} e^{\lambda x}$. Near $x = 0$ the factor $x^{-\gamma}$ dominates; near $x = \infty$ the factor $e^{\lambda x}$ does. Hence:

- $\int_0^c s(x)\, \mathrm{d}x = +\infty \iff \gamma \geq 1 \iff r - qE \geq \sigma^2/2$;
- $\int_c^\infty s(x)\, \mathrm{d}x = +\infty$ always, by the exponential factor.

Lemma 2.2 then gives:

Teorema 3.1 (Persistence). *The SDE (9) admits a unique strong solution in $(0, \infty)$ for all $t \geq 0$ (both boundaries non-attainable) if and only if*

$$r - qE \;>\; \frac{\sigma^2}{2}. \tag{13}$$

In particular, in the absence of harvesting the condition reduces to $r > \sigma^2/2$.

Step 3 (stationary density). Substituting (10) and (11) into (6):

$$p^*(x) \;=\; C\,x^{\gamma-2}\,e^{-\lambda x}, \qquad x > 0. \tag{14}$$

This is the density of a Gamma distribution with shape parameter $\eta = \gamma - 1$ and rate parameter λ. It is integrable on $(0,\infty)$ if and only if $\gamma > 1$, recovering condition (13): *the analytic criterion (Feller) and the probabilistic criterion (integrability of p^*) coincide.*

Proposición 3.2 (Moments). *Under* (13), $X_\infty \sim \mathrm{Gamma}(\gamma-1, 1/\lambda)$ *with*

$$\mathbb{E}[X_\infty] \;=\; K\Big(1-\tfrac{qE}{r}\Big) - \tfrac{K\sigma^2}{2r}, \qquad \mathrm{Var}(X_\infty) \;=\; \tfrac{\sigma^2K^2}{2r^2}\Big(r - qE - \tfrac{\sigma^2}{2}\Big). \tag{15}$$

The stationary mean decomposes as $\mathbb{E}[X_\infty] = X_2^* - K\sigma^2/(2r)$: noise *biases mean abundance downward* relative to the deterministic equilibrium, and the correction is independent of the harvesting effort E.

3.3. Stochastic maximum sustainable yield

The expected stationary yield is $R(E) = qE\,\mathbb{E}[X_\infty]$. Maximizing over E using (15):

$$E^{\mathrm{stoch}}_{\mathrm{MSY}} \;=\; \frac{r}{2q} - \frac{\sigma^2}{4q}, \qquad Y^{\mathrm{stoch}}_{\mathrm{MSY}} \;=\; \frac{K\big(r-\sigma^2/2\big)^2}{4r}, \qquad X^{*,\mathrm{stoch}}_{\mathrm{MSY}} \;=\; \frac{K}{2} - \frac{K\sigma^2}{4r}. \tag{16}$$

Table 2 compares the deterministic and stochastic results; note that every stochastic correction is a function of the single dimensionless parameter σ^2/r, and all of them vanish in the limit $\sigma \to 0$.

Cuadro 2: Deterministic vs. stochastic comparison for (7)–(9).

Quantity	**Deterministic** ($\sigma = 0$)	**Stochastic** ($\sigma > 0$)
Persistence	$qE < r$	$qE < r - \sigma^2/2$
Equilibrium state	point $K(1 - qE/r)$	density $\mathrm{Gamma}(\gamma-1, 1/\lambda)$
Mean abundance	$K(1 - qE/r)$	$K(1 - qE/r) - K\sigma^2/(2r)$
MSY effort	$r/(2q)$	$r/(2q) - \sigma^2/(4q)$
MSY yield	$rK/4$	$K(r - \sigma^2/2)^2/(4r)$
Stock at MSY	$K/2$	$K/2 - K\sigma^2/(4r)$

3.4. Why the control structure matters: quota vs. effort

The dictionary clarifies why apparently similar harvesting policies produce qualitatively different outcomes. Under *constant-effort* harvesting qEX, the drift near $x = 0$ behaves as $(r - qE)x$, linear in x, which gives $\kappa(x) \sim \gamma/x$ and the boundary can be non-attainable under (13). Under *constant-quota* harvesting $\alpha > 0$, instead, the drift is $\mu(x) = rx(1 - x/K) - \alpha$ and the coefficient $\kappa(x) = 2\mu(x)/(\sigma^2x^2)$ contains a term $-2\alpha/(\sigma^2x^2)$, non-integrable near zero. This reverses the sign of the scale integral in (5): the boundary $x = 0$ becomes *attainable and absorbing* for every $\alpha > 0$, and extinction occurs with probability one [2, §2]. The structural difference — constant per-capita rate vs. a per-capita rate α/x that blows up — is detected directly in $\kappa(x)$, with no new techniques required.

4. Discussion

The procedure laid out in Section 2 is fully general for one-dimensional diffusions with regular coefficients: once (μ, Σ) are identified, all qualitative questions (global existence, persistence, ergodicity, stationary moments) reduce to the analysis of the integral $\int^x \kappa$. The application of

Section 3 is just one of many: the same scheme produces the log-normal stationary distribution for the Gompertz model with multiplicative-noise harvesting, the Gamma distribution for the Cox–Ingersoll–Ross process ($\Sigma(x) = \sigma\sqrt{x}$, $\mu(x) = \kappa(\theta - x)$), and the normal distribution for the Ornstein–Uhlenbeck process. In each case, the only thing that changes is the result of integrating κ; the structure of the analysis is the same.

The dictionary has precise limits. For jump-diffusion processes, the scale function ceases to capture hitting probabilities (the process may "jump over" a boundary), and the Fokker–Planck equation acquires integral terms [4]. For piecewise-deterministic Markov processes there is no diffusion at all, and ergodicity must be established via Harris recurrence theory [6]. Nevertheless, the *architecture* of the analysis (evolution equation → stationary state → optimal control via HJB) survives these generalizations; what changes are the technical tools, not the conceptual framework.

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